Natural Language Models and Interfaces

BSc Artificial Intelligence

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Institute for Logic, Language, and Computation

2019, week 6

Data sparsity

We have so far dealt with categorical models using tabular CPDs

- we've encountered problems for maximum likelihood estimation due to data sparsity
- large n-grams lead to large tables $O(v^n)$
- even the emission distributions of HMMs had to be smoothed
- smoothing techniques can be rather brittle

Are there principled ways to tackle data sparsity?

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Let's check a running example based on *sentiment classification*

Example: sentiment classification

How can we identify whether a sentence is positive or negative towards a subject?

- This movie is slow and repetitive, clearly the direction was careless and the production cheap.
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If y is the sentiment of some text x, then we would like to compute $P_{Y|X}(y|x)$, but

- x is very sparse!
- how could we possibly parameterise it?

Feature functions

Suppose we identify a number of words which typically express sentiment, let's call them *features*

Class	**Features (or attributes)**
Negative	cheap, slow, repetitive, careless, awful, bad, ...
Positive	funny, spiced, interesting, awesome, good, ...

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Then for an example
$x =$ 'This film is fun though shy on the action' with sentiment $y = +$, let us retain only the sentiment words:

- $\langle$'fun', 'shy', 'though', 'action'$\rangle =$ $\mathrm{sentwords}($'This film is fun though shy on the action'$)$

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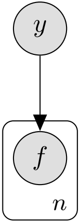
- $\langle$'fun', 'shy', 'though', 'action'$\rangle =$ $\mathrm{sentwords}($'This film is fun though shy on the action'$)$
- we will denote this by a random pair $(Y, \langle F_1, \ldots, F_n \rangle)$

Naive Bayes Classifiers

In a naive Bayes classifier, we assume the *class* y generates the features $\langle f_1, \ldots, f_n \rangle$



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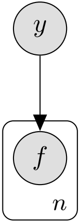

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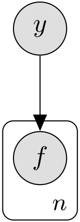

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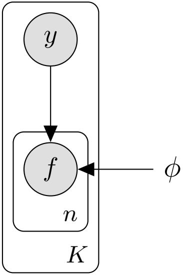

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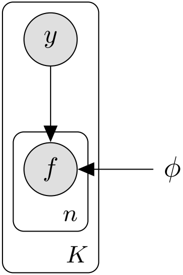

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The generative story for each training instance:

$$Y \sim \mathcal{U}(1/2)$$

$$\text{for } i = 1, \ldots, n$$

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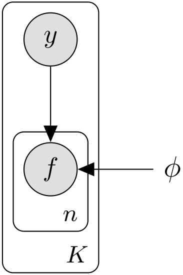

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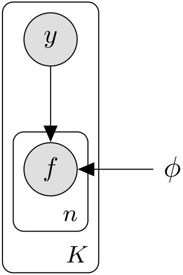

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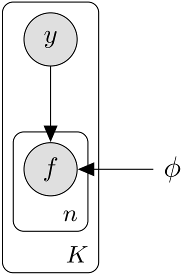

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Naive Bayes Binary Classification: MLE

The log-likelihood of the data is proportional to

$$\log P(\mathcal{D}|\phi) = \prod_{k=1}^{K} P_{Y|F_1^n}(y_k, \langle f_1^{(k)}, \ldots, f_{n_k}^{(k)} \rangle | \phi)$$

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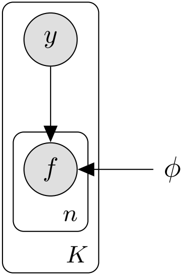

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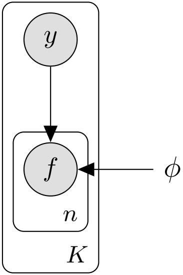

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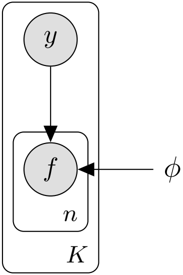

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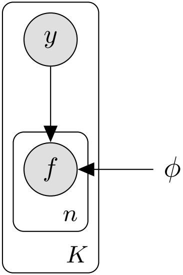

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$$\phi_f^{(y)} = \frac{\text{count}_{YF}(y, f)}{\text{count}_{Y}(y)}$$

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For some new example with features $\langle f_1, \ldots, f_n \rangle$, we can predict its class easily by solving a maximisation problem

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Problem with Naive Bayes classification

Instead of computing $P_{Y|X}(y|x)$

- we compute $P_{Y|F_1^n}(y|f_1^n)$ using some features f_1^n of x
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- but an increase in feature space, e.g. $O(v^3)$ for trigram features, leads to problems for parameter estimation

Conditioning on high-dimensional data

The problem is that we only know tabular CPDs

If Y takes on values in $\mathcal{Y}$ and X takes on values in $\mathcal{X}$, tabular CPDs associate a parameter $\theta_y^{(x)}$ with each outcome y in context x

$$P_{Y|X}(y|x) = \mathrm{Cat}(y|\theta^{(x)}) = \theta_y^{(x)}$$

This can only work if $|\mathcal{Y}|$ and $|\mathcal{X}|$ are relatively small

- representation cost $O(|\mathcal{Y}| \times |\mathcal{X}|)$

If x is itself very high-dimensional (e.g. a sentence), this cannot possibly work (as in this case $\mathcal{X} \subseteq \Sigma^*$)

Are there other representations to CPDs?

Let's focus on the case where x is high-dimensional and y is binary
How can we assign a value to $P_{Y|X}(y|x)$?

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- we can then make the probability value $P_{Y|X}(y|x)$ depend functionally on $f(x, y)$
- but we need to make sure that $0 \leq P_{Y|X}(y|x) \leq 1$ and that $\sum_y P_{Y|X}(y|x) = 1$

Feature function

An example of a binary feature function

Y X	1 This film is fun and full of action	0 This film is full of boring action
action_+	1	0
action_-	0	1
boring_+	0	0
boring_-	0	1
full_+	1	0
full_-	0	1
fun_+	1	0
fun_-	0	0

Table: Feature function: $f : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}^D$

- binary feature functions map the input to a D-dimensional vector of feature indicators

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  No! Because $w^\top f(x, y)$ can be negative and
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Suppose we have a D-dimensional real vector $w \in \mathbb{R}^D$

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Logistic regression - MLE

We model the conditional using logistic regression

$$P_{Y|X}(y|x) = \frac{\exp\Big(w^\top f(x, y)\Big)}{\sum_{y' \in \mathcal{Y}} \exp\Big(w^\top f(x, y)\Big)}$$

Then with a dataset $\mathcal{D} = \{(x^{(k)}, y^{(k)})\}_{k=1}^N$ of i.i.d. observations, what's the maximum likelihood estimate for $w \in \mathbb{R}^D$?

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recall where "count and divide" comes from

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We look for w that is solution to $\boldsymbol{\nabla}_w \mathcal{L}(w|\mathcal{D}) = 0$ where

$$\mathcal{L}(w|\mathcal{D}) = \sum_{k=1}^{N} \underbrace{\log P_{Y|X}(y^{(k)}|x^{(k)}, w)}_{\ell(w|x^{(k)}, y^{(k)})}$$

is the log-likelihood function

Let's start with a single training instance

The log-likelihood function gets a contribution $\ell(w|x,y) = \log P_{Y|X}(y|x,w)$ from each training instance

Let's expand ℓ slightly

$$\log P_{Y|X}(y|x,w) = \log \frac{\exp\Big(w^\top f(x,y)\Big)}{\sum_{y' \in \mathcal{Y}} \exp(w^\top f(x,y'))}$$

$$=$$

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&= w^\top f(x, y) - \log \underbrace{\sum_{y' \in \mathcal{Y}} \exp\Big(w^\top f(x, y')\Big)}_{\mathcal{Z}(x|w)}
\end{aligned}
$$

We need $\nabla_w \log P_{Y|X}(y|x, w)$ but let's first take the gradient of the partition function $\mathcal{Z}(x|w)$

Gradient of partition function

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&= \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x,w) f(x,y) = \mathbb{E}[f(x,Y)]
\end{aligned}$$

Putting everything together

We know

- $\nabla_w \mathcal{Z}(x|w) = \sum_{y \in \mathcal{Y}} \exp\Big(w^\top f(x,y)\Big) f(x,y)$
- $\ell(w|x,y) = w^\top f(x,y) - \log \mathcal{Z}(x|w)$
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Gradient descent

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Then

$$\nabla_w \ell(w|x,y) = f(x,y) - \mathbb{E}[f(x,Y)]$$

There is no closed-form solution to $\nabla_w \ell(w|x,y) = 0$, but there is an iterative algorithm that converges to the solution

$$w^{(t+1)} = w^{(t)} + \gamma \nabla_{w^{(t)}} \ell(w^{(t)}|x,y)$$

$\gamma > 0$ is called the *learning rate* (a hyperparameter)

Gradient descent

Maximum likelihood estimation for logistic regression

We look for w that is solution to $\nabla_w \mathcal{L}(w|\mathcal{D}) = 0$ where

$$\mathcal{L}(w|\mathcal{D}) = \sum_{k=1}^{N} \underbrace{\log P_{Y|X}(y^{(k)}|x^{(k)}, w)}_{\ell(w|x^{(k)}, y^{(k)})}$$

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Stochastic gradient ascent

We can use unbiased *stochastic gradient estimates* instead of the full gradient

$$w^{(t+1)} = w^{(t)} + \gamma^{(t)} \frac{M}{N} \sum_{s=1}^{M} \boldsymbol{\nabla}_{w^{(t)}} \ell(w^{(t)} | x^{(s)}, y^{(s)})$$

where $S \sim \mathcal{U}(1/N)$ selects training instances uniformly at random

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where $S \sim \mathcal{U}(1/N)$ selects training instances uniformly at random
The learning rate $\gamma > 0$ must follow a particular schedule, e.g.

$$\gamma^{(t)} = \frac{\gamma^{(t)}}{1 + \gamma^{(0)}\alpha t}$$

where the initial learning rate $\gamma^{(0)} > 0$ and the rate of decay $\alpha > 0$ are hyperparameters

Regularisation

To avoid overfitting to training instances, we place a penalty on awkwardly large weights, our objective becomes

$$\operatorname*{argmax}_{w \in \mathbb{R}^D} \; \mathcal{L}(w|\mathcal{D}) - \frac{\lambda}{2}||w||^2$$

where λ is the weight of the L_2 regulariser

Our gradient becomes

$$\begin{aligned} \boldsymbol{\nabla}_w \left(\mathcal{L}(w|\mathcal{D}) - \lambda ||w||^2 \right) &= \boldsymbol{\nabla}_w \mathcal{L}(w|\mathcal{D}) - \frac{\lambda}{2} \boldsymbol{\nabla}_w \sum_{d=1}^{D} w_d^2 \\ &= \boldsymbol{\nabla}_w \mathcal{L}(w|\mathcal{D}) - \lambda \sum_{d=1}^{D} w_d \end{aligned}$$

Summary

Logistic regression allows us to express statistical dependencies between two variables through a finite set of features

- we can directly model a conditional probability using rich features of a high-dimensional conditioning context (this is called a *logistic cpd*)
- without the need for the strong independence assumptions
- we have to estimate D parameters (the weights of a log-linear model)
- MLE does not have a closed-form solution, but gradient ascent gives us an iterative algorithm

Next class we will see how this can be used for various tasks
e.g. sentiment classification, language identification,
POS tagging, language modelling

References I